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On the Monodromy Groups of Riemann Surfaces of Genus Zero

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It is known that every finite group G can appear as the monodromy group of some Riemann surface of genus $\geqslant 0$. The fact that symmetric groups of all orders can appear as monodromy groups of Riemann surfaces of genus zero is a long-standing one. In this paper, a further search has been made in order to determine which finite groups G can and cannot appear as monodromy groups of Riemann surfaces of genus zero. It has been shown, on the one hand, that every alternating group, the simple group PSL(2, 7) and all cyclic and dihedral groups can appear as such monodromy groups by using a right coset representation of each with respect to a particular subgroup. It has been shown, on the other hand, that the quaternion group, the generalized quaternion group of order 16, a non-Abelian group of order 27 with specified presentation as well as every finite direct product $\mathsf{X}_{i=1}^{n} Z_{m_i}$ with $m_i \mid m_{i+1}$ and $n > 1$ of cyclic groups of order >4 can never appear as such monodromy groups using right coset representations with respect to subgroups. The examples cited above suggest the conjecture that all simple groups can appear as monodromy groups (right coset representations with respect to subgroups being employed to determine the monodromy group) of Riemann surfaces of genus zero and that noncyclic, solvable groups cannot appear unless they are extensions of degree 2 of a cyclic group.

It is known [4, 8] that every finite group appears as the monodromy group of a Riemann surface R which, in turn, is the Galois group G of an irreducible algebraic equation $P(w, z) = 0$ for w with coefficients in the field of rational functions of z with complex coefficients. It is also known from the existence theorems of uniformization theory that every topologically defined Riemann surface (defined by a set of permutations; see Hurwitz [7]) is indeed the Riemann surface of a function $w(z)$ defined by an equation $P(w, z) = 0$.

We shall investigate the question: For which groups G can R be of genus zero? If the Riemann surface of $P(w, z) = 0$ has genus zero, we know further that we can uniformize the corresponding algebraic curve by rational functions by putting $w = R_1(t)$, $z = R_2(t)$ with rational R_1, R_2 and with the parameter t expressible as a rational function of w and z.

If we insist on Riemann surfaces which are Galois coverings of the plane or,

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in other words, for which the degree of P in w equals the order of G, it seems that R can be of genus zero only if G has a faithful representation as a group of Möbius transformations. This case has been well investigated. The most complex and most famous subcase is the one for which G is the alternating group on five symbols. An explicit form for the equation $P(w, z) = 0$ of 60th degree has been obtained by F. Klein and can be found in [3, pp. 53–54, 98]. However, we may also consider the cases for which $P(w, z) = 0$ is not a normal equation for w. In this case, it can be considered as a resolvent of lower degree for a normal equation with Galois group G.

The examples computed below suggest the conjecture that all simple groups can appear as monodromy groups of Riemann surfaces of genus zero when the coset representation of each with respect to some subgroup is used and that noncyclic, solvable groups cannot appear using such representations unless they are extensions of degree two of a cyclic group.

The explicit and well-known construction of Hurwitz [7] shows that a topological Riemann surface R_l with n sheets and branch points P_λ, $\lambda = 1, 2, \ldots, l$, (i.e., an n-sheeted ramified covering of the two-sphere S^2 with l points P_λ removed and ramification points over the points P_λ only) is defined by the following data: the points P_λ and a set of permutations π_λ acting on n symbols (the sheets) in such a manner that the π_λ generate a transitive permutation group $M(R_l)$ and satisfy the relation

$$\pi_1 \pi_2 \cdots \pi_l = 1,$$

where 1 denotes the identical permutation. Letting h denote the total number of cycles (including 1-cycles) occurring in all of the π_λ, then $\beta = nl - h$ is the branching number of R_l. The group $M(R_l)$ generated by the π_λ is called the *ordinary Riemann monodromy group of the surface* R_l.

In the present paper, we begin to classify which finite groups G can and cannot appear as monodromy groups $M(R_l)$ of Riemann surfaces R_l of genus zero. In order to effect such a classification, we must determine whether or not we can:

(I)

(a) faithfully represent G as a transitive permutation group $\pi(G)$ on n symbols in such a way that

(b) the product of the generating permutations $\pi(g_\nu)$, corresponding to a suitably selected set of generators g_ν of G, in some appropriately arranged order is the identical permutation 1, and

(c) $\beta = 2n - 2$, $h = n(l - 2) + 2$, where β is obtained by summing all $\sum_{i=1}^{m}(l_i - 1)$ with l_i representing the length of each cycle in the disjoint product of cycles of each $\pi(g_\nu)$ in (b) and observing that m varies with each ν are simultaneously satisfied.

Such a representation can be accomplished by using the right cosets of the various subgroups H in G as follows: Let G be a finite group and let H be a subgroup. For each $g \in G$, there is a permutation of the set of right cosets of H

$$\pi(g) = \binom{Hx}{Hxg}, \qquad x \in G, \tag{1}$$

so that $g \mapsto \pi(g)$ is a representation of G as a transitive permutation group on the distinct right cosets of H and $\pi(g)$ fixes H if and only if $g \in H$. Conversely, suppose that $g \mapsto \pi(g)$ is a representation of G as a transitive permutation group P on a set of elements S. If s_1 is a particular element of S, then the g's such that $\pi(g)$ fixes s_1 form a subgroup H of G and the elements of S can be put into a one-to-one correspondence with the right cosets of H so that P is isomorphic as a permutation group to the group of permutations $\pi(G)$ in (1). Furthermore, in the representation (1), the elements mapped onto the identity form the largest normal subgroup of G contained in H. Thus, (1) is a faithful representation if and only if H contains no normal subgroups (including H) of G greater than the identity. An immediate implication is that the only faithful transitive representation of an Abelian group is the regular representation. (For results, see [5].)

If a finite group G can appear as the monodromy group (obtained by using right coset representations of G with respect to subgroups H) of some Riemann surface of genus zero, we will say below that it is of *type* $MR(0)$. If it cannot appear as such a monodromy group (i.e., again using right coset representations), we will say that it is of *type* $NMR(0)$.

The fact that all symmetric groups S_n, $n \geqslant 2$, on n symbols are of type $MR(0)$ is a long-standing one. This fact becomes apparent if we simply let $(1\ 2), (1\ 3), \ldots, (1\ n)$ be the generators of a faithful representation of S_n of order $n!$ on its maximal nonnormal subgroup of order $(n-1)!$ and then observe that $\prod_{i=2}^{n} (1\ i)(1\ i) = 1$ and hence that $\beta = 2(n-1)$. Thus (I) holds and the conclusion is immediate.

Furthermore, since the regular representation is the only faithful, transitive representation of a cyclic group, we also find trivially:

THEOREM 1. *Cyclic groups C_n of all orders n, $n \geqslant 2$, are of type $MR(0)$.*

Proof. Let a be a generator of C_n and let $a \mapsto (1\ 2 \cdots n)$ be a faithful representation of C_n. Since $(1\ 2 \cdots n)(1\ 2 \cdots n)^{-1} = 1$ and hence $\beta = 2(n-1)$, then (I) holds and the claim is valid. ∎

We prove additionally that a large class of simple groups can be typed as $MR(0)$ in the following:

THEOREM 2. *Alternating groups A_n of every possible order $n!/2$, $n \geqslant 3$ are of type $MR(0)$.*

Proof. Since $A_3 \approx C_3$, Theorem 1 is sufficient to give the result when $n = 3$.

If $n > 3$, let

$$s = (3\ 4 \cdots n), \qquad t = (1\ 2\ 3) \qquad (n \text{ odd})$$

or

$$s = (1\ 2)(3\ 4 \cdots n), \qquad t = (1\ 2\ 3) \qquad (n \text{ even})$$

be generators of a faithful representation of A_n of order $n!/2$ on its maximal nonnormal subgroup of order $(n-1)!/2$. When n is odd,

$$(3\ 4 \cdots n)(1\ 2\ 3)(1\ 2 \cdots n)^{-1} = 1 \qquad \text{with} \quad \beta = 2n - 2.$$

When n is even,

$$[(1\ 2)(3\ 4 \cdots n)](1\ 2\ 3)(1\ 3\ 4 \cdots n)^{-1} = 1 \qquad \text{with} \quad \beta = 2n - 2.$$

Thus, (I) holds for $n > 3$ as well. ∎

The next conclusion allows us to admit still another simple group to those of type $MR(0)$.

THEOREM 3. PSL(2, 7), *the projective special linear group of degree* 2 *over the field of integers* mod 7, *having presentation*

$$\left\langle A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad A^7 = (AB)^3 = B^2 = (A^2BA^4B)^3 = I \right\rangle,$$

is of type $MR(0)$.

Proof. Let H be the cyclic subgroup of PSL(2, 7) of order 7 generated by A, i.e., let

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}, \nu = 0, 1, \ldots, 6 \right\}.$$

If $C = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2, 7)$, then the cosets of H in PSL(2, 7) assume the form

$$HC = \left\{ \begin{pmatrix} a & b \\ \nu a + c & \nu b + d \end{pmatrix}, \nu = 0, 1, \ldots, 6 \right\}.$$

Furthermore, by (1), every $D \in \mathrm{PSL}(2, 7) \mapsto \left(\begin{smallmatrix} HC \\ HCD \end{smallmatrix}\right)$. If D has order 2, 3, or 4, then $HC \neq HCD$. For, suppose the contrary. Then, $CD = (\pm F)C$ for some $F \neq I \in H$ implying that $D = C^{-1}(\pm F)C$ and thus contradicting the fact that all elements in the same conjugacy class must have the same order. Therefore, all cosets are moved upon right multiplication by elements of order 2, 3, or 4. If D is the element $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ of order 7, then it can easily but tediously be shown that exactly three cosets stay fixed while those remaining move under the action of (1).

Since $i_G(H) = 24$, the number of sheets of the corresponding surface is 24 and β must be 46. However, by the above comments, we observe that elements of order 2, 3, 4 are, respectively, mapped by (1) into permutations consisting of twelve 2-cycles, eight 3-cycles, six 4-cycles with respective contributions of 12, 16, 18 to β while A is mapped to a permutation consisting of three 7-cycles with a contribution of 18 to β.

Now, since A, B, and $C = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ generate PSL(2, 7) and since $\pi(CBA) = \pi(I)$ with a computed β by I(c) of 46, (I) applies and PSL(2, 7) is of type $MR(0)$ as claimed. (Note: Although the conclusion was formulated in terms of the particular cyclic subgroup H of order 7, it could have equally well been formulated in terms of various other subgroups of PSL(2, 7).) ∎

THEOREM 4. *Dihedral groups D_n of orders $2n$, $n \in Z^+$, generated by the two elements s, t satisfying the relations $s^n = 1$, $t^2 = 1$, $tst = s^{-1}$, are of type $MR(0)$.*

Proof. Note first that t and st, both of order 2, will generate each D_n.

Let n be odd and $\geqslant 3$. Let $H = \{1, t\}$ be the nonnormal subgroup of D_n with $i_G(H) = n$ so that $\beta = 2n - 2$. The coset decomposition of H in G is given as

$$Hs^i = \{s^i, s^{n-i}t\}, \qquad i = 0, 1,..., n-1.$$

Label these cosets respectively 1, 2,..., n. Then, by (1),

$$t \mapsto \pi(t) = (1)(2\ n)(3\ n-1) \cdots (n+1/2 \quad n+3/2)$$

and

$$st \mapsto \pi(st) = (1\ n)(2\ n-1) \cdots (n-1/2\ n+3/2)(n+1/2).$$

Since $[\pi(st)]^2[\pi(t)]^2 = \pi(1)$ with calculated β to be $2[(n-1)/2] + 2[(n-1)/2] = 2n-2$, (I) holds. Thus, each D_n, n odd and $\geqslant 3$ is of type $MR(0)$.

Let n be even and $\geqslant 4$. Let H be as above. Then,

$$t \mapsto \pi(t) = (1)(2\ n)(3\ n-1) \cdots (n/2\ n+4/2)(n+2/2)$$

and

$$st \mapsto \pi(st) = (1\ n)(2\ n-1) \cdots (n/2\ n+2/2).$$

Again, $[\pi(st)]^2[\pi(t)]^2 = \pi(1)$ with $\beta = 2n-2$. Thus, each D_n, n even and $\geqslant 4$ is of type $MR(0)$.

We argue separately that D_2 is of type $MR(0)$. Since D_2 is Abelian, we can represent it faithfully and transitively only by using its subgroup 1 of index 4 with

$$\pi(s) = (1\ 2)(3\ 4), \qquad \pi(t) = (1\ 3)(2\ 4), \qquad \pi(st) = (1\ 4)(2\ 3),$$

and with $\pi(s)\,\pi(t)\,\pi(st) = \pi(1)$ and a calculated β of 6. Once again, (I) holds and D_2 is of type $MR(0)$.

Since $D_1 \approx C_2$, Theorem 1 applies and D_1 is also of type $MR(0)$.

Piecing the above paragraphs together, we conclude that every D_n, $n \in Z^+$, is of type $MR(0)$. ∎

Recognizing that certain dihedral groups are noncyclic and solvable of prime power order 2^n, we question whether all noncyclic, solvable groups of prime power order p^n, $n \geqslant 2$, are of type $MR(0)$. The answer is seen to be negative by the straightforward:

THEOREM 5. *Each of the following groups,*

(i) *the quaternion group of order* 8 *with presentation*

$$Q_2 = \langle a, b;\ a^4 = 1,\ a^2 = (ab)^2 = b^2\rangle,$$

(ii) *the generalized quaternion group of order* 16 *with presentation*

$$Q_4 = \langle a, b;\ a^4 = b^2,\ bab^{-1} = a^{-1}\rangle,$$

(iii) *the non-Abelian group of order* 27 *having the presentation*

$$T = \langle a, b;\ a^9 = b^3 = 1,\ b^{-1}ab = a^4\rangle,$$

is of type $NMR(0)$.

Proof. (i) Every subgroup of Q_2 is a normal subgroup so that Q_2 can only be faithfully represented on the subgroup 1 with $i_G(1) = 8$ and $\beta = 14$. Since two elements of order 4 are needed in order to generate Q_2, the corresponding permutational generators under (1) will each contribute 6 to β implying that (b) and (c) of (I) are incompatible in this instance thus barring Q_2's appearance.

(ii) The subgroup $\{1, a^4\}$ of order 2 is normal in Q_4 and is contained in every subgroup of order 4. Thus, Q_4 can be faithfully represented only by using 1. The elements of order 2, 4, 8 are, respectively, mapped by (1) into permutations consisting of eight 2-cycles, four 4-cycles, two 8-cycles which, in turn, contribute 8, 12, 14 to β which must be 30 by theory. Again, (b) and (c) of (I) are incompatible and Q_4 is of type $NMR(0)$.

(iii) The three subgroups $\{b^i a^3\}$ $(i = 1, 2)$, $\{b\}$, of index 9 in T, and the trivial subgroup 1 of index 27 in T are the only subgroups which will provide faithful representations. However, any of these choices leads again to the incompatibility of (b) and (c) of (I) so that T is ruled out from appearing as well.

In Theorem 1, we established that cyclic groups of all orders are of type $MR(0)$. We now prove the result that all Abelian groups other than cyclic with presentation $\times_{i=1}^{n} Z_{m_i}$, $m_i \mid m_{i+1}$, $n > 1$, $m_i \neq 1$, are of type $NMR(0)$.

THEOREM 6. *All direct products* $\times_{i=1}^{n} Z_{m_i}$, $m_i \mid m_{i+1}$, $n > 1$, $m_i \neq 1$, *of cyclic groups are of type* $NMR(0)$ *with the exception of the Klein four group* $Z_2 \times Z_2$ *which is of type* $MR(0)$.

Proof. A minimal set of generators for $G = \times_{i=1}^{n} Z_{m_i}$ consists of n elements and can be taken to be the set

$$\{k_1 = (g_1, 1,..., 1), k_2 = (1, g_2, ..., 1),..., k_n = (1, 1,..., g_n\}, \tag{2}$$

where g_i is a generator of Z_{m_i}. G can be faithfully represented only on the subgroup 1 of index $m_1 m_2 \cdots m_n$ in G so that the number of sheets of the corresponding surface must be $\prod_{i=1}^{n} m_i$ and β must be $2[\prod_{i=1}^{n} m_i] - 2$.

We prove now that (b) and (c) of (I) are inconsistent, which will yield the desired conclusion. In other words, we show below that β_C obtained by the counting argument of I(c) is greater than $\beta_T = 2[\prod_{i=1}^{n} m_i] - 2$ as known by the Riemann–Hurwitz relation for any selected set of generators $\pi(a_i)$ of $\pi(G)$ satisfying $\prod \pi(a_i) = 1$, i.e., we show that for any such set of $\pi(a_i)$ that

$$\beta_C \geqslant n\left[\prod_{i=1}^{n} m_i\right] - \sum_{i=1}^{n} m_1 m_2 \cdots \hat{m}_i \cdots m_n > 2\left[\prod_{i=1}^{n} m_i\right] - 2 = \beta_T, \tag{3}$$

where $\wedge$ indicates that m_i is to be deleted from the shown product for each i. Equivalently, we show that

$$n > 2 + \left[\sum_{i=1}^{n} 1/m_i\right] - 2\Big/\prod_{i=1}^{n} m_i \tag{4}$$

or that

$$n > 2 + \sum_{i=1}^{n} 1/m_i \tag{5}$$

upon division by $\prod_{i=1}^{n} m_i$.

Let $m_i = 2$ for all $i = 1,..., n$. Then, (5) $\Leftrightarrow n > 4$ with implication that if all $m_i \geqslant 2$ and if n is fixed >4, then (3) is valid and (b) and (c) of (I) cannot be consistent. Therefore, all direct products $\times_{i=1}^{n} Z_{m_i}$, $m_i \geqslant 2$, $n > 4$, are of type $NMR(0)$.

It remains to investigate the instances $n = 4, 3, 2$.

When $n = 4$, (4) becomes

$$4 > 2 + \left[\sum_{i=1}^{4} 1/m_i\right] - 2\Big/\prod_{i=1}^{4} m_i . \tag{6}$$

Letting $m_i = 2$ for each i, (6) $\Leftrightarrow 4 > 31/8$, a validity implying the truth of (3) for all $m_i \geqslant 2$. We conclude that all products $\mathsf{X}_{i=1}^{4} Z_{m_i}$ are of type $NMR(0)$ provided that all $m_i \geqslant 2$.

When $n = 3$ and $m_3 \geqslant m_1, m_2$, (3) reduces to showing that

$$\beta_C \geqslant 4\left[\prod_{i=1}^{3} m_i\right] - 2m_1m_2 - m_2m_3 - m_1m_3 > 2\left[\prod_{i=1}^{3} m_i\right] - 2 = \beta_T , \tag{7}$$

which is equivalent to showing that

$$2 > 2/m_3 + 1/m_1 + 1/m_2 - 2\Big/\prod_{i=1}^{3} m_i . \tag{8}$$

Let $m_i = 2$ for each i. Then (8) $\Leftrightarrow 2 > 7/8$ with the implication that (7) holds for all $m_i \geqslant 2$. Thus, each $\mathsf{X}_{i=1}^{3} Z_{m_i}$, $m_i \geqslant 2$, is of type $NMR(0)$.

When $n = 2$ and $m_2 \geqslant m_1$, (3) reduces to proving

$$\beta_C \geqslant 2\left[\prod_{i=1}^{2} m_i\right] - \left[\sum_{i=1}^{2} \hat{m}_i\right] + m_1[m_2 - 1] > 2\left[\prod_{i=1}^{2} m_i\right] - 2 = \beta_T \tag{9}$$

or the equivalent

$$1 > 2/m_2 + 1/m_1 - 2\Big/\prod_{i=1}^{2} m_i . \tag{10}$$

If $m_1 = m_2 = 3$, then (10) $\Leftrightarrow 1 > 7/9$ and therefore (9) will be true for all $m_i \geqslant 3$, $i = 1, 2$. If $m_1 = m_2 = 2$, then both (9) and (10) become equalities. If $m_1 = 2$ and $m_2 = 4$, then (10) $\Leftrightarrow 1 > 3/4$ so that (9) will be true for all $m_1 = 2$ and $m_2 \geqslant 4$. Conclude that every direct product $\mathsf{X}_{i=1}^{2} Z_{m_i}$, $m_i \geqslant 2$, is of type $NMR(0)$ with the exception of $Z_2 \times Z_2$, which is of type $MR(0)$.

Piecing the subcases proved in the above paragraphs together, the proof is complete. ∎

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